

The elliptic quantum algebra $U_{q,p}(\widehat{sl_N})$ and its vertex operators

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Abstract

We construct a realization of the elliptic quantum algebra $U_{q,p}(\widehat{sl_N})$ for any given level k in terms of free boson fields and their twisted partners. It can be considered as the elliptic deformation of the Wakimoto realization of the quantum affine algebra $U_q(\widehat{sl_N})$. We also construct a family of screening currents, which commute with the currents of $U_{q,p}(\widehat{sl_N})$ up to total q -differences. And we give explicit twisted expressions for the type *I* and the type *II* vertex operators of $U_{q,p}(\widehat{sl_N})$ by twisting the known results of the type *I* vertex operators of the quantum affine algebra $U_q(\widehat{sl_N})$ and the new results of the type *II* vertex operators of $U_q(\widehat{sl_N})$ we obtained in this paper.

1 Introduction

Infinite-dimensional symmetries, such as the Virasoro algebra (W -algebra in more general) and affine Lie algebras play central roles in the two-dimensional conformal field theories (2D CFT) [1]. For the non-conformal (off-critical) integrable theories, their roles are taken over by the so called quantum algebras. From the algebraic point of view, there are three kinds of quantum algebras,

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according to different exchange properties, which are nominated as rational, trigonometric and elliptic quantum algebras respectively. The quantum algebras of the former two kinds could be regarded as certain degenerate cases of the latter one. For example, the quantum affine algebras (trigonometric), which are also known as the quantum group [2, 3], and the Yangian double [4] with central (rational) can be obtained as a certain limited case of the elliptic quantum algebras. Various versions of elliptic quantum algebras, also called as elliptic quantum groups [5, 6, 7] have been introduced to understand elliptic face models of statistical mechanics, and in their semiclassical limit, CFT of Wess-Zumino-Witten (WZW) models on tori. Their roles are similar to the Kac-Moody algebras in WZW models. From the Hopf algebra point of view, the elliptic quantum groups are nothing but quantum affine algebras equipped with a co-product different from the original one by a certain kind of twisting, so they can be viewed as quasi-Hopf algebras in the sense of Drinfeld [8]. They have two types which correspond to different types of integrable models: the vertex type $A_{q,p}(\widehat{sl_N})$ and the face type $B_{q,\lambda}(\mathcal{G})$, where \mathcal{G} is a Kac-Moody algebra associated with a symmetrizable generalized Cartan matrix [9]. The former is closely related to vertex models, for example, the XYZ model, or equivalently, the eight vertex model in the principal regime [10]; while some face models, such as the Andrew-Baxter-Forrester (ABF) models [11] which are ‘solid-on-solid’ (SOS) face models, possess symmetries corresponding to the face type elliptic algebras $B_{q,\lambda}(\mathcal{G})$.

In mathematics, it is natural to study these algebraic objects’ structures and their representations. In physical applications, their representations are also required. The standard scheme to study integrable models in field theories or statistical mechanics is solving the following basic problems: to diagonalize the given Hamiltonian and then to compute the correlation functions. Usually, it is quite difficult to solve such problems directly. It has been indicated that the algebraic analysis method is an extremely powerful tool in studying solvable lattice models, especially in deriving the correlation functions. This method is based on the infinite dimensional quantum group symmetry possessed by a solvable lattice model and the representation theory of such symmetry. This algebraic method could be viewed as the quantum version of the powerful Inverse Scattering Method [12]. See [13] for a review on it. As a result, if one expects to perform algebraic analysis over the above two types of elliptic lattice models, he should first study the corresponding elliptic quantum groups and their representations.

It is of special interest for the algebra of the intertwining operators in the WZW model. It was derived by Knizhnik and Zamolodchikov that the matrix coefficients of the intertwining operators for the WZW model satisfy certain holonomic differential equations, i.e., the Knizhnik-Zamolodchikov(KZ) equation [14]. In [15], for quantum affine algebra, the authors defined q -deformed vertex operators as certain intertwining operators and showed that they satisfied some holonomic difference equations called the quantum KZ(qKZ) equations. So it is also expected that

the representations of the elliptic quantum algebras are helpful in constructing the elliptic type solutions of quantum Knizhnik-Zamolodchikov-Bernard(qKZB) equation, which is a higher genus extension of the qKZ equation [16].

At classical level, there are various models of representations for the current algebras and each of them is of significance in certain applications. Here, we just mention two of them: the Wakimoto construction (free field realization) [17, 18, 19], and the parafermion realization [20, 21, 22]. Recently, the explicit description of free field realizations of current algebras were given in [23, 24, 25]. In [26, 27], the XXZ model in the anti-ferromagnetic regime was solved by applying the level one representation theory of the quantum affine algebra $U_q(\widehat{sl_2})$. In studying a higher spin extension of the XXZ model, the realizations of $U_q(\widehat{sl_2})$ at level $k > 1$ are required, and they were constructed by several authors, such as the Wakimoto realization in [28] and the parafermion realizations in [29, 30]. Furthermore, in [31], free field realization of $U_q(\widehat{sl_N})$ with arbitrary level $k \geq 1$ was given, and it plays a central role in understanding the higher rank extension of the XXZ model. The Wakimoto construction is also a powerful way to study the integrable massive field theories [32]. In practice, free field realization, which is an infinite dimensional extension of the Heisenberg algebra, is quite an effective and useful approach to study complicated algebraic structures and their representations. The level k free field representation of Yangian double $DY_h(sl_2)$ and applications in physical problems were discussed in [33, 34]. The level one free field realization of the Yangian double with central $DY_h(sl_N)$ was constructed in [35], while the level k representation of $DY_h(sl_N)$ and $DY_h(gl_N)$ were given in [36]. It should also be remarked that the Yangian double with central $DY_h(\widehat{sl_2})$ is the symmetry possessed by the Sine-Gordon model, which is the field theory limit of the restricted SOS (RSOS) model [37, 38].

It is first noticed by Lukyanov and Pugai [39] that a symmetry of the RSOS model is generated by the q -deformation of the Virasoro algebra (q -Virasoro algebra). The free field realizations of screening currents and vertex operators enable them to analyze the structure of the highest weight representation of the q -Virasoro algebra. And the screening currents they constructed satisfy an elliptic deformation of $U_q(\widehat{sl_2})$ at level one, which is called the elliptic algebra $U_{q,p}(\widehat{sl_2})$. In [39] the elliptic algebra is obtained by twisting the Cartan current. In some sense, we say that the elliptic algebra at level one governs the structure of the q -Virasoro algebra. It seems true that it also holds for their higher rank extensions. So following this approach and the above mentioned expectations, it is important to obtain the realizations of the elliptic quantum algebras. In fact, for studying the RSOS model and its higher spin extension (i.e. the k -fusion RSOS model), the representations of $U_{q,p}(\widehat{sl_2})$ with any given level k have been presented in [40] and [41]. They are different from each other. The former can be viewed as the elliptic version of the parafermionic realization, which is obtained by twisting the parafermionic realization of the quantum affine algebra $U_q(\widehat{sl_2})$; and the latter is the elliptic deformation of the Wakimoto realization. The elliptic algebra $U_{q,p}(\widehat{sl_2})$

is actually the Drinfeld realization of $B_{q,\lambda}(\widehat{sl_2})$ showed in [42]. Furthermore, in order to study a higher rank extension of the RSOS model, we should construct the realizations of $U_{q,p}(\widehat{sl_N})$. It can be viewed as the Drinfeld realization of the face type elliptic algebra $B_{q,\lambda}(\widehat{sl_N})$ showed in [42, 43]. However only in the level-one case, the parafermion realization of it was given in [43]. And it can not be extended to the higher level k , although parafermion theory is important in physics [20, 21, 22] and in mathematics [44]. The realizations of [40, 43] are based on the facts that in $\widehat{su(2)}_k$ case, the parafermions are decoupled from the Cartan current, while in $\widehat{su(N)}_1$ case, the parafermions become trivial (i.e. identity operator). In fact, the bosonization of non-local currents for higher rank and higher-level algebras is a huge project even in the classical level. So if one wants to deal with the elliptic quantum algebra of higher rank through bosonization of the non-local currents, it will not be a practical way. In this paper we will introduce a new way to construct the free field representation of the higher rank algebra $U_{q,p}(\widehat{sl_N})$. It is the higher rank generalization of the construction in [41]. And our construction could be viewed as a twisted version of the quantum semi-infinite flag manifolds [18].

In free fields approach, there are two necessary ingredients that one has to discuss: screening currents and vertex operators (VOs). They all play crucial roles in calculating correlation functions and investigating the irreducible representations. The screening currents commute or anti-commute with the currents of $U_{q,p}(\widehat{sl_N})$ up to a total q -difference of some fields. And for this algebra, there are two kinds of VOs with distinct physical applications: the type *I* VOs and the type *II* VOs. The former is a local operator which describes the operation of adding one lattice site, and the formula of the correlation functions can be expressed as traces of the product of these operators over irreducible representation space; while the latter plays the role of particle creation or annihilation operators. In this paper, we also construct the free field realization of these two important objects. In fact, they are all obtained by twisting the corresponding ones of the quantum affine algebra $U_q(\widehat{sl_N})$. In order to do that, we have to construct the type *II* VOs of $U_q(\widehat{sl_N})$ which have never been given before. In fact, even for the classical affine algebra, the type *II* VOs of it are unknown.

In this paper, in section 2 we define the Drinfeld realization of $U_{q,p}(\widehat{sl_N})$ as a certain tensor product of the quantum affine algebra $U_q(\widehat{sl_N})$ and a Heisenberg algebra, which is different from the ones given in [42, 43]. With this definition it is more convenient to construct the free field representation of $U_{q,p}(\widehat{sl_N})$ with given level k . And in section 3 we will present the construction in two steps. In section 4 a series of screening currents of $U_{q,p}(\widehat{sl_N})$ are given. In section 5 the explicit expressions of the type *II* VOs of $U_q(\widehat{sl_N})$ and the two types VOs of $U_{q,p}(\widehat{sl_N})$ are presented.

2 The elliptic quantum algebra $U_{q,p}(\widehat{sl_N})$

There are two types of the elliptic quantum algebras: the face type and the vertex type. Here we only consider the face type elliptic algebra $U_{q,p}(\widehat{sl_N})$, which can be viewed as the Drinfeld realization of the face type elliptic quantum group $B_{q,\lambda}(\widehat{sl_N})$. Usually, we can also consider it as the tensor product of the quantum affine algebra $U_q(\widehat{sl_N})$ and a Heisenberg algebra. In this section, we will first review the definition of $U_q(\widehat{sl_N})$; then we will define the elliptic currents of it; lastly, we give the definition of the elliptic algebra $U_{q,p}(\widehat{sl_N})$. Throughout this paper, we fix a complex number $q \neq 0$, $|q| < 1$.

2.1 The quantum affine algebra $U_q(\widehat{sl_N})$

In this subsection, for convenience, we give a review of the definition of $U_q(\widehat{sl_N})$. We will use the standard symbol $[n]$:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}},$$

and let $A = (a_{ij})_{1 \leq i, j \leq N-1}$ be the Cartan matrix of sl_N . The dual Coxeter number of it is denoted by h^\vee and $h^\vee = N$.

Definition 1 . $U_q(\widehat{sl_N})$ is the associative algebra over \mathbb{C} with Drinfeld generators H_n^i ($n \in \mathbb{Z} - \{0\}$), $e_n^{\pm, i}$ ($n \in \mathbb{Z}$), h_i ($i = 1, \dots, N-1$) and the central element c satisfying the following defining relations:

$$[h_i, H_n^j] = 0, \quad [h_i, e_n^{\pm, j}] = \pm a_{ij} e_n^{\pm, j} \quad (2.1)$$

$$[H_n^i, H_m^j] = \frac{[a_{ij}n][cn]}{n} \delta_{n+m, 0}, \quad (2.2)$$

$$[H_n^i, e_m^{\pm, j}] = \pm \frac{[a_{ij}n]}{n} q^{\mp \frac{cn}{2}} e_{n+m}^{\pm, j}, \quad (2.3)$$

$$[e_n^{\pm, i}, e_m^{\mp, j}] = \frac{\delta^{ij}}{q - q^{-1}} (q^{\frac{cn}{2}(n-m)} \psi_{+, n+m}^i - q^{-\frac{cn}{2}(n-m)} \psi_{-, n+m}^i), \quad (2.4)$$

$$[e_{n+1}^{\pm, i}, e_m^{\pm, j}]_{q^{\pm a_{ij}}} + [e_{m+1}^{\pm, j}, e_n^{\pm, i}]_{q^{\pm a_{ij}}} = 0, \quad (2.5)$$

$$[e_n^{\pm, i}, e_m^{\pm, j}] = 0 \quad \text{for } a_{ij} = 0, \quad (2.6)$$

$$[e_n^{\pm, i}, [e_m^{\pm, i}, e_l^{\pm, j}]_{q^{\pm 1}}]_{q^{\pm 1}} + [e_m^{\pm, i}, [e_n^{\pm, i}, e_l^{\pm, j}]_{q^{\pm 1}}]_{q^{\pm 1}} = 0 \quad \text{for } a_{ij} = -1, \quad (2.7)$$

where $\psi_{\pm, n}^i$ are defined by

$$\sum_{n \in \mathbb{Z}} \psi_{\pm, n}^i z^{-n} = q^{\pm h_i} \exp \left(\pm (q - q^{-1}) \sum_{\pm n > 0} H_n^i z^{-n} \right),$$

and the symbol $[A, B]_x$ for $x \in \mathbb{C}$ denotes $AB - xBA$.

If we introduce the generating functions $\psi_{\pm}^i(z)$ and $e^{\pm, i}(z)$ ($i = 1, \dots, N-1$) as

$$\psi_{\pm}^i(z) = \sum_{n \in \mathbb{Z}} \psi_{\pm, n}^i z^{-n}, \quad e^{\pm, i}(z) = \sum_{n \in \mathbb{Z}} e_n^{\pm, i} z^{-n-1},$$

which are called the Drinfeld currents of $U_q(\widehat{sl_N})$. In terms of them, the above defining relations (2.1)-(2.7) can be recast as

$$[\psi_{\pm}^i(z), \psi_{\pm}^j(w)] = 0, \quad (2.8)$$

$$\begin{aligned} (z - q^{a_{ij}-c}w)(z - q^{-a_{ij}+c}w)\psi_+^i(z)\psi_-^j(w) \\ = (z - q^{a_{ij}+c}w)(z - q^{-a_{ij}-c}w)\psi_-^j(w)\psi_+^i(z), \end{aligned} \quad (2.9)$$

$$(z - q^{\pm(a_{ij}-\frac{c}{2})}w)\psi_+^i(z)e^{\pm, j}(w) = (q^{\pm a_{ij}}z - q^{\mp \frac{c}{2}}w)e^{\pm, j}(w)\psi_+^i(z), \quad (2.10)$$

$$(z - q^{\pm(a_{ij}-\frac{c}{2})}w)e^{\pm, j}(z)\psi_-^j(w) = (q^{\pm a_{ij}}z - q^{\mp \frac{c}{2}}w)\psi_-^j(w)e^{\pm, j}(z), \quad (2.11)$$

$$[e^{+, i}(z), e^{-, j}(w)] = \frac{\delta^{ij}}{(q - q^{-1})zw} \left(\delta(q^c w/z)\psi_+^i(q^{\frac{c}{2}}w) - \delta(q^{-c} w/z)\psi_-^j(q^{-\frac{c}{2}}w) \right), \quad (2.12)$$

$$(z - q^{\pm a_{ij}}w)e^{\pm, i}(z)e^{\pm, j}(w) = (q^{\pm a_{ij}}z - w)e^{\pm, j}(w)e^{\pm, i}(z), \quad (2.13)$$

$$e^{\pm, i}(z)e^{\pm, j}(w) = e^{\pm, j}(w)e^{\pm, i}(z) \quad \text{for } a_{ij} = 0, \quad (2.14)$$

$$\begin{aligned} e^{\pm, i}(z_1)e^{\pm, i}(z_2)e^{\pm, j}(w) - [2]e^{\pm, i}(z_1)e^{\pm, j}(w)e^{\pm, i}(z_2) \\ + e^{\pm, j}(w)e^{\pm, i}(z_1)e^{\pm, i}(z_2) + (\text{replacement} : z_1 \leftrightarrow z_2) = 0 \quad \text{for } a_{ij} = -1, \end{aligned} \quad (2.15)$$

where $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$.

2.2 The elliptic algebra $U_{q,p}(\widehat{sl_N})$

The elliptic algebra $U_{q,p}(\widehat{sl_N})$ can be considered as the tensor product of the elliptic currents of $U_q(\widehat{sl_N})$ and a Heisenberg algebra [42]. We first give the elliptic currents of $U_q(\widehat{sl_N})$. A pair of parameters p and p^* will be used:

$$p = q^{2r}, \quad p^* = q^{2r^*} = pq^{-2c} \quad (r^* = r - c; \quad r, r^* \in \mathbb{R}_{>0}).$$

Let us define the currents $D_i^{\pm}(z; r, r^*) \in U_q(\widehat{sl_N})$ ($i = 1, \dots, N-1$) depending on r and r^* as

$$\begin{aligned} D_i^+(z; r, r^*) &= \exp \left(\sum_{n>0} \frac{1}{[r^*n]} H_{-n}^i q^{(r^*+\frac{c}{2})n} z^n \right), \\ D_i^-(z; r, r^*) &= \exp \left(- \sum_{n>0} \frac{1}{[rn]} H_n^i q^{(r-\frac{c}{2})n} z^{-n} \right), \end{aligned}$$

which are different from the ones in [42, 43] by a power of q . Using them we can define the ‘dressed’ currents $\Psi_i^\pm(z)$, $e_i(z)$ and $f_i(z)$ ($i = 1, \dots, N-1$) as:

$$\begin{aligned}\Psi_i^+(z) &= D_i^+(q^{\frac{c}{2}}z; r, r^*)\psi_+^i(z)D_i^-(q^{-\frac{c}{2}}z; r, r^*), \\ \Psi_i^-(z) &= D_i^+(q^{-\frac{c}{2}}z; r, r^*)\psi_-^i(z)D_i^-(q^{\frac{c}{2}}z; r, r^*), \\ e_i(z) &= D_i^+(z; r, r^*)e^{+,i}(z), \\ f_i(z) &= e^{-,i}(z)D_i^-(z; r, r^*).\end{aligned}$$

Obviously these currents all depend on the parameter p . Moreover, applying (2.8)-(2.15) we have the following proposition by direct calculation:

Proposition 1 . *The fields $\Psi_i^\pm(z)$, $e_i(z)$ and $f_i(z)$ ($i = 1, \dots, N-1$) defined above satisfy the following elliptic commutation relations:*

$$\Psi_i^\pm(z)\Psi_j^\pm(w) = \frac{\Theta_p(q^{-a_{ij}}\frac{z}{w})\Theta_{p^*}(q^{a_{ij}}\frac{z}{w})}{\Theta_p(q^{a_{ij}}\frac{z}{w})\Theta_{p^*}(q^{-a_{ij}}\frac{z}{w})}\Psi_j^\pm(w)\Psi_i^\pm(z), \quad (2.16)$$

$$\Psi_i^+(z)\Psi_j^-(w) = \frac{\Theta_p(pq^{-a_{ij}-c}\frac{z}{w})\Theta_{p^*}(p^*q^{a_{ij}+c}\frac{z}{w})}{\Theta_p(pq^{a_{ij}-c}\frac{z}{w})\Theta_{p^*}(p^*q^{-a_{ij}+c}\frac{z}{w})}\Psi_j^-(w)\Psi_i^+(z), \quad (2.17)$$

$$\Psi_i^\pm(z)e_j(w) = q^{-a_{ij}}\frac{\Theta_{p^*}(q^{\pm\frac{c}{2}+a_{ij}}\frac{z}{w})}{\Theta_{p^*}(q^{\pm\frac{c}{2}-a_{ij}}\frac{z}{w})}e_j(w)\Psi_i^\pm(z), \quad (2.18)$$

$$\Psi_i^\pm(z)f_j(w) = q^{a_{ij}}\frac{\Theta_p(q^{\mp\frac{c}{2}-a_{ij}}\frac{z}{w})}{\Theta_p(q^{\mp\frac{c}{2}+a_{ij}}\frac{z}{w})}f_j(w)\Psi_i^\pm(z), \quad (2.19)$$

$$[e_i(z), f_j(w)] = \frac{\delta^{ij}}{(q - q^{-1})zw}\left(\delta(q^{-c}\frac{z}{w})\Psi_i^+(q^{\frac{c}{2}}w) - \delta(q^c\frac{z}{w})\Psi_i^-(q^{-\frac{c}{2}}w)\right), \quad (2.20)$$

$$e_i(z)e_j(w) = q^{-a_{ij}}\frac{\Theta_{p^*}(q^{a_{ij}}\frac{z}{w})}{\Theta_{p^*}(q^{-a_{ij}}\frac{z}{w})}e_j(w)e_i(z), \quad (2.21)$$

$$f_i(z)f_j(w) = q^{a_{ij}}\frac{\Theta_p(q^{-a_{ij}}\frac{z}{w})}{\Theta_p(q^{a_{ij}}\frac{z}{w})}f_j(w)f_i(z), \quad (2.22)$$

$$\begin{aligned}&\frac{(p^*q^{\frac{2}{z_1}}; p^*)_\infty}{(p^*q^{-\frac{2}{z_1}}; p^*)_\infty}\left\{e_j(w)e_i(z_1)e_i(z_2) - [2]\frac{(p^*q^{\frac{z_1}{w}}; p^*)_\infty(p^*q^{-1}\frac{w}{z_1}; p^*)_\infty}{(p^*q^{-1}\frac{z_1}{w}; p^*)_\infty(p^*q^{\frac{w}{z_1}}; p^*)_\infty}e_i(z_1)e_j(w)e_i(z_2)\right. \\ &\quad \left. + \frac{(p^*q^{\frac{z_1}{w}}; p^*)_\infty(p^*q^{-1}\frac{w}{z_1}; p^*)_\infty}{(p^*q^{-1}\frac{z_1}{w}; p^*)_\infty(p^*q^{\frac{w}{z_1}}; p^*)_\infty}\frac{(p^*q^{\frac{z_2}{w}}; p^*)_\infty(p^*q^{-1}\frac{w}{z_2}; p^*)_\infty}{(p^*q^{-1}\frac{z_2}{w}; p^*)_\infty(p^*q^{\frac{w}{z_2}}; p^*)_\infty}e_i(z_1)e_i(z_2)e_j(w)\right\} \\ &\quad + (\text{replacement} : z_1 \leftrightarrow z_2) = 0 \quad \text{for } |i - j| \leq 1, \quad (2.23)\end{aligned}$$

$$\begin{aligned}&\frac{(pq^{-\frac{2}{z_1}}; p)_\infty}{(pq^{\frac{2}{z_1}}; p)_\infty}\left\{f_j(w)f_i(z_1)f_i(z_2) - [2]\frac{(pq^{\frac{w}{z_1}}; p)_\infty(pq^{-1}\frac{z_1}{w}; p)_\infty}{(pq^{-1}\frac{z_1}{w}; p)_\infty(pq^{\frac{z_1}{w}}; p)_\infty}f_i(z_1)f_j(w)f_i(z_2)\right. \\ &\quad \left. + \frac{(pq^{\frac{w}{z_1}}; p)_\infty(pq^{-1}\frac{z_1}{w}; p)_\infty}{(pq^{-1}\frac{z_1}{w}; p)_\infty(pq^{\frac{z_1}{w}}; p)_\infty}\frac{(pq^{\frac{w}{z_2}}; p)_\infty(pq^{-1}\frac{z_2}{w}; p)_\infty}{(pq^{-1}\frac{z_2}{w}; p)_\infty(pq^{\frac{z_2}{w}}; p)_\infty}f_i(z_1)f_i(z_2)f_j(w)\right\} \\ &\quad + (\text{replacement} : z_1 \leftrightarrow z_2) = 0 \quad \text{for } |i - j| \leq 1, \quad (2.24)\end{aligned}$$

where we use the elliptic theta function $\Theta_t(z)$ for any parameter $t = q^{2\nu}$ ($\nu \in \mathbb{C}$) defined as

$$\Theta_t(z) = (z; t)_\infty (tz^{-1}; t)_\infty (t; t)_\infty,$$

in which

$$(z; t_1, \dots, t_k)_\infty = \prod_{n_1, \dots, n_k \geq 0} (1 - z t_1^{n_1} \dots t_k^{n_k}).$$

Here these ‘dressed’ currents $\Psi_i^\pm(z)$, $e_i(z)$ and $f_i(z)$ ($i = 1, \dots, N-1$) are called the elliptic currents of $U_q(\widehat{sl_N})$ since they obey the above elliptic commutation relations.

Next, we need a set of Heisenberg algebras generated by P_i , Q_i ($i = 1, \dots, N-1$) with

$$[P_i, Q_j] = -\frac{a_{ij}}{2},$$

to add nice periodicity properties to the elliptic exchange relations (2.16)-(2.24). And the Heisenberg algebras commute with $U_q(\widehat{sl_N})$. For convenience, the following parametrization will be used in the following sections:

$$\begin{aligned} q &= e^{-\pi i / r\tau}, \\ p &= e^{-2\pi i / \tau}, \quad p^* = e^{-2\pi i / \tau^*} \\ z &= q^{2u} = e^{-2\pi i u / r\tau}. \end{aligned}$$

With them, we can further define the currents $H_i^\pm(u)$, $E_i(u)$ and $F_i(u)$ ($i = 1, \dots, N-1$) as follows:

$$\begin{aligned} H_i^\pm(u) &= \Psi_i^\pm(z) e^{2Q_i} q^{\mp h_i} (q^{\pm(r-\frac{\epsilon}{2})} z)^{\frac{(h_i+P_i-1)}{r} - \frac{(P_i-1)}{r^*}}, \\ E_i(u) &= e_i(z) e^{2Q_i} z^{-\frac{(P_i-1)}{r^*}}, \\ F_i(u) &= f_i(z) z^{\frac{(h_i+P_i-1)}{r}} \end{aligned}$$

They are actually the tensor product of elliptic currents $\Psi_i^\pm(z)$, $e_i(z)$ and $f_i(z)$ with the Heisenberg algebras. And to distinguish them from the elliptic currents, we call them the total currents. It should be noted that the choice of the zero-modes in $H_i^\pm(u)$, $E_i(u)$ and $F_i(u)$ are different from the ones given in [42, 43]. Our choice makes our construction of the free field realization of $U_{q,p}(\widehat{sl_N})$ more convenience. Now the definition of $U_{q,p}(\widehat{sl_N})$ can be stated explicitly as:

Definition 2 . The elliptic algebra $U_{q,p}(\widehat{sl_N})$ is isomorphic to the associative algebra over \mathbb{C} generated by $H_i^\pm(u)$, $E_i(u)$ and $F_i(u)$ ($i = 1, \dots, N-1$) with the following defining relations:

$$H_i^\pm(u) H_j^\pm(v) = \frac{\theta_r(u-v-\frac{a_{ij}}{2}) \theta_{r^*}(u-v+\frac{a_{ij}}{2})}{\theta_r(u-v+\frac{a_{ij}}{2}) \theta_{r^*}(u-v-\frac{a_{ij}}{2})} H_j^\pm(v) H_i^\pm(u), \quad (2.25)$$

$$H_i^+(u)H_j^-(v) = \frac{\theta_r(u-v-\frac{c}{2}-\frac{a_{ij}}{2})}{\theta_r(u-v-\frac{c}{2}+\frac{a_{ij}}{2})} \frac{\theta_{r^*}(u-v+\frac{c}{2}+\frac{a_{ij}}{2})}{\theta_{r^*}(u-v+\frac{c}{2}-\frac{a_{ij}}{2})} H_j^-(v)H_i^+(u), \quad (2.26)$$

$$H_i^\pm(u)E_j(v) = \frac{\theta_{r^*}(u-v\pm\frac{c}{4}+\frac{a_{ij}}{2})}{\theta_{r^*}(u-v\pm\frac{c}{4}-\frac{a_{ij}}{2})} E_j(v)H_i^\pm(u), \quad (2.27)$$

$$H_i^\pm(u)F_j(v) = \frac{\theta_r(u-v\mp\frac{c}{4}-\frac{a_{ij}}{2})}{\theta_r(u-v\mp\frac{c}{4}+\frac{a_{ij}}{2})} F_j(v)H_i^\pm(u), \quad (2.28)$$

$$[E_i(u), F_j(v)] = \frac{\delta_{ij}}{(q-q^{-1})z\omega} \left(\delta(u-v-\frac{c}{2})H_i^+(u-\frac{c}{4}) - \delta(u-v+\frac{c}{2})H_i^-(v-\frac{c}{4}) \right), \quad (2.29)$$

$$E_i(u)E_j(v) = \frac{\theta_{r^*}(u-v+\frac{a_{ij}}{2})}{\theta_{r^*}(u-v-\frac{a_{ij}}{2})} E_j(v)E_i(u), \quad (2.30)$$

$$F_i(u)F_j(v) = \frac{\theta_r(u-v-\frac{a_{ij}}{2})}{\theta_r(u-v+\frac{a_{ij}}{2})} F_j(v)F_i(u), \quad (2.31)$$

$$\begin{aligned} & z_1^{-\frac{2}{r^*}} \frac{(p^*q^{\frac{2z_2}{z_1}}; p^*)_\infty}{(p^*q^{-2\frac{z_2}{z_1}}; p^*)_\infty} \left\{ E_j(u)E_i(u_1)E_i(u_2) \right. \\ & \quad - [2] \left(\frac{z}{z_1} \right)^{\frac{a_{ij}}{r^*}} \frac{(p^*q^{\frac{a_{ij}z}{z_1}}; p^*)_\infty (p^*q^{-a_{ij}\frac{z_1}{z}}; p^*)_\infty}{(p^*q^{-a_{ij}\frac{z}{z_1}}; p^*)_\infty (p^*q^{\frac{a_{ij}z_1}{z}}; p^*)_\infty} E_i(u_1)E_j(u)E_i(u_2) \\ & \quad + \left(\frac{z}{z_1} \right)^{\frac{a_{ij}}{r^*}} \left(\frac{z}{z_2} \right)^{\frac{a_{ij}}{r^*}} \frac{(p^*q^{\frac{a_{ij}z}{z_1}}; p^*)_\infty (p^*q^{-a_{ij}\frac{z_1}{z}}; p^*)_\infty}{(p^*q^{-a_{ij}\frac{z}{z_1}}; p^*)_\infty (p^*q^{\frac{a_{ij}z_1}{z}}; p^*)_\infty} \\ & \quad \times \frac{(p^*q^{\frac{a_{ij}z}{z_2}}; p^*)_\infty (p^*q^{-a_{ij}\frac{z_2}{z}}; p^*)_\infty}{(p^*q^{-a_{ij}\frac{z}{z_2}}; p^*)_\infty (p^*q^{\frac{a_{ij}z_2}{z}}; p^*)_\infty} E_i(u_1)E_i(u_2)E_j(u) \Big\} \\ & \quad + (\text{replacement} : z_1 \leftrightarrow z_2) = 0 \quad \text{for } |i-j| \leq 1, \end{aligned} \quad (2.32)$$

$$\begin{aligned} & z_1^{\frac{2}{r}} \frac{(pq^{-2\frac{z_2}{z_1}}; p)_\infty}{(pq^{\frac{2z_2}{z_1}}; p)_\infty} \left\{ F_j(u)F_i(u_1)F_i(u_2) \right. \\ & \quad - [2] \left(\frac{z}{z_1} \right)^{-\frac{a_{ij}}{r}} \frac{(pq^{-a_{ij}\frac{z}{z_1}}; p)_\infty (pq^{\frac{a_{ij}z_1}{z}}; p)_\infty}{(pq^{\frac{a_{ij}z}{z_1}}; p)_\infty (pq^{-a_{ij}\frac{z_1}{z}}; p)_\infty} F_i(u_1)F_j(u)F_i(u_2) \\ & \quad + \left(\frac{z}{z_1} \right)^{-\frac{a_{ij}}{r}} \left(\frac{z}{z_2} \right)^{-\frac{a_{ij}}{r}} \frac{(pq^{-a_{ij}\frac{z}{z_1}}; p)_\infty (pq^{\frac{a_{ij}z_1}{z}}; p)_\infty}{(pq^{\frac{a_{ij}z}{z_1}}; p)_\infty (pq^{-a_{ij}\frac{z_1}{z}}; p)_\infty} \\ & \quad \times \frac{(pq^{-a_{ij}\frac{z}{z_2}}; p)_\infty (pq^{\frac{a_{ij}z_2}{z}}; p)_\infty}{(pq^{\frac{a_{ij}z}{z_2}}; p)_\infty (pq^{-a_{ij}\frac{z_2}{z}}; p)_\infty} F_i(u_1)F_i(u_2)F_j(u) \Big\} \\ & \quad + (\text{replacement} : z_1 \leftrightarrow z_2) = 0 \quad \text{for } |i-j| \leq 1, \end{aligned} \quad (2.33)$$

where the notation of the Jacobi theta functions $\theta_v(u)$ for $v \in \mathbb{C}$ are used,

$$\theta_v(u) = q^{\frac{u^2}{v}-u} \frac{\Theta_{q^{2v}}(q^{2u})}{(q^{2v}; q^{2v})_\infty^3}.$$

Note that we have used the parametrization $z = q^{2u}$, $w = q^{2v}$ and $z_i = q^{2u_i}$ ($i = 1, 2$) in the above expressions. In the following, we will use this parametrization without mentioning them if they are not confused. It is easy to see that the above relations (2.25)-(2.33) have good periodicity properties because of the quasi-periodicity property of the Jacobi theta functions, such as

$$\theta_r(u + r) = -\theta_r(u), \quad \theta_r(u + r\tau) = -e^{-\pi\tau i - 2\pi i u/r} \theta_r(u)$$

and similar relations hold for $\theta_{r^*}(u)$ with r replaced by r^* .

3 Free field realization of $U_{q,p}(\widehat{sl_N})$

The level k representation of $U_{q,p}(\widehat{sl_N})$ has not been given before. Although the free field realization of it in level 1 was given in [43], it can not be generalized to the higher level case. In this section, by using a new method, we will construct a free boson realization of $U_{q,p}(\widehat{sl_N})$ with given level k . This method has been used to construct a free field realization of $U_{q,p}(\widehat{sl_2})_k$ in [41]. Here we will show that it can be generalized to the higher rank case. The method is to twist the level k Wakimoto realization of $U_q(\widehat{sl_N})$ by constructing some ‘twisting’ currents. We will first fix some conventions and review the Wakimoto realization of $U_q(\widehat{sl_N})$ in [31]; then we will give our construction in two steps: the first one is the bosonization of the elliptic currents of $U_q(\widehat{sl_N})$; and the second one is the free boson realization of the total currents.

3.1 Notations

We introduce a quantum Heisenberg algebra $\mathcal{H}_{q,k}$ with the generators: a_n^i, p_a^i, q_a^i for $1 \leq i \leq N-1$; $b_n^{ij}, p_b^{ij}, q_b^{ij}$ and $c_n^{ij}, p_c^{ij}, q_c^{ij}$ for $1 \leq i < j \leq N$, where $n \in \mathbb{Z}_{\neq 0}$, and the defining relations are as follows:

$$\begin{aligned} [a_n^i, a_m^j] &= \frac{[(k + h^\vee)n][a_{ij}n]}{n} \delta_{n+m,0}, \quad [p_a^i, q_a^j] = a_{ij}(k + h^\vee), \\ [b_n^{ij}, b_m^{j'j'}] &= -\frac{[n]^2}{n} \delta^{ii'} \delta^{jj'} \delta_{n+m,0}, \quad [p_b^{ij}, q_b^{i'j'}] = -\delta^{ii'} \delta^{jj'}, \\ [c_n^{ij}, c_m^{j'j'}] &= \frac{[n]^2}{n} \delta^{ii'} \delta^{jj'} \delta_{n+m,0}, \quad [p_c^{ij}, q_c^{i'j'}] = \delta^{ii'} \delta^{jj'} \end{aligned}$$

and the others vanish. Using them, we set the generating functions $a^i(z; \alpha)$ for $\alpha \in \mathbb{C}$ and $a_\pm^i(z)$ ($1 \leq i \leq N-1$) by:

$$\begin{aligned} a^i(z; \alpha) &= - \sum_{n \neq 0} \frac{a_n^i}{[n]} q^{-\alpha|n|} z^{-n} + q_a^i + p_a^i \ln z, \\ a_\pm^i(z) &= \pm (q - q^{-1}) \sum_{n > 0} a_{\pm n}^i z^{\mp n} + p_a^i \ln q \end{aligned}$$

and $a^i(z; 0) \equiv a^i(z)$ for simplicity. Similarly, the generating functions $b^{i,j}(z; \alpha)$, $b_{\pm}^{i,j}(z)$ and $c^{i,j}(z; \alpha)$, $c_{\pm}^{i,j}(z)$ for $1 \leq i < j \leq N$ can also be given. These generating functions can be viewed as some free bosonic fields, if we consider the generators of $\mathcal{H}_{q,k}$ as the modes of $N^2 - 1$ free bosons: a^i ($1 \leq i \leq N - 1$), b^{ij} and c^{ij} ($1 \leq i < j \leq N$). We also define the completion $\widehat{\mathcal{H}_{q,k}}$ of $\mathcal{H}_{q,k}$ as:

$$\widehat{\mathcal{H}_{q,k}} = \lim_{\leftarrow} \mathcal{H}_{q,k}/I_n, \quad n > 0,$$

where I_n is the left ideal of $\mathcal{H}_{q,k}$ generated by all the polynomials in $\{a_m^i (1 \leq i \leq N - 1), b_m^{ij}$ and $c_m^{ij} (1 \leq i < j \leq N): m > 0\}$ of degree greater than or equal to n (here we set $\deg(a_m^i) = \deg(b_m^{ij}) = \deg(c_m^{ij}) = m$). Normal order prescription $:$ is set by moving $a_n^i (n > 0)$ and p_a^i to the right, while moving $a_n^i (n < 0)$ and q_a^i to the left. For example,

$$: \exp(a^i(z)) := \exp\left(-\sum_{n < 0} \frac{a_n^i}{[n]} z^{-n}\right) e^{q_a^i} z^{p_a^i} \exp\left(-\sum_{n > 0} \frac{a_n^i}{[n]} z^{-n}\right).$$

In terms of the above free bosonic fields, we can define a homomorphism $h_{q,k}$ from the algebra $U_q(\widehat{sl_N})$ to $\widehat{\mathcal{H}_{q,k}}$. It is defined on the generators by:

$$\begin{aligned} h_{q,k}(\psi_{\pm}^i(z)) =: & \exp\left(\sum_{j=1}^i (b_{\pm}^{j,i+1}(q^{\pm(\frac{k}{2}+j-1)}z) - b_{\pm}^{j,i}(q^{\pm(\frac{k}{2}+j)}z)) + a_{\pm}^i(q^{\pm\frac{k}{2}}z) \right. \\ & \left. + \sum_{j=i+1}^N (b_{\pm}^{i,j}(q^{\pm(\frac{k}{2}+j)}z) - b_{\pm}^{i+1,j}(q^{\pm(\frac{k}{2}+j-1)}z))\right) :, \end{aligned} \quad (3.1)$$

$$\begin{aligned} h_{q,k}(e^{+,i}(z)) = & \frac{-1}{(q - q^{-1})z} \sum_{j=1}^i : \exp((b + c)^{j,i}(q^{j-1}z)) \\ & \times \left(\exp(b_+^{j,i+1}(q^{j-1}z) - (b + c)^{j,i+1}(q^jz)) \right. \\ & \left. - \exp(b_-^{j,i+1}(q^{j-1}z) - (b + c)^{j,i+1}(q^{j-2}z)) \right) \\ & \times \exp\left(\sum_{l=1}^{j-1} (b_+^{l,i+1}(q^{l-1}z) - b_+^{l,i}(q^l z))\right) :, \end{aligned} \quad (3.2)$$

$$\begin{aligned} h_{q,k}(e^{-,i}(z)) = & \frac{-1}{(q - q^{-1})z} \left(\sum_{j=1}^{i-1} : \exp((b + c)^{j,i+1}(q^{-(k+j)}z)) \right. \\ & \times \left(\exp(-b_-^{j,i}(q^{-(k+j)}z) - (b + c)^{j,i}(q^{-(k+j-1)}z)) \right. \\ & \left. - \exp(-b_+^{j,i}(q^{-(k+j)}z) - (b + c)^{j,i}(q^{-(k+j+1)}z)) \right) \\ & \times \exp\left(\sum_{l=j+1}^i (b_-^{l,i+1}(q^{-(k+l-1)}z) - b_-^{l,i}(q^{-(k+l)}z)) \right. \\ & \left. + a_-^i(q^{-\frac{k+h}{2}}z) + \sum_{l=i+1}^N (b_-^{i,l}(q^{-(k+l)}z) - b_-^{i+1,l}(q^{-(k+l-1)}z)) \right) : \end{aligned}$$

$$\begin{aligned}
& + : \exp\left((b+c)^{i,i+1}(q^{-(k+i)}z)\right) \\
& \quad \times \exp\left(a_-^i(q^{-\frac{k+h^\vee}{2}}z) + \sum_{l=i+1}^N (b_-^{i,l}(q^{-(k+l)}z) - b_-^{i+1,l}(q^{-(k+l-1)}z))\right) : \\
& - : \exp\left((b+c)^{i,i+1}(q^{k+i}z)\right) \\
& \quad \times \exp\left(a_+^i(q^{\frac{k+h^\vee}{2}}z) + \sum_{l=i+1}^N (b_+^{i,l}(q^{k+l}z) - b_+^{i+1,l}(q^{k+l-1}z))\right) : \\
& - \sum_{j=i+2}^N : \exp\left((b+c)^{i,j}(q^{k+j-1}z)\right) \\
& \quad \times \left(\exp(b_+^{i+1,j}(q^{k+j-1}z) - (b+c)^{i+1,j}(q^{k+j}z)) \right. \\
& \quad \left. - \exp(b_-^{i+1,j}(q^{k+j-1}z) - (b+c)^{i+1,j}(q^{k+j-2}z)) \right) \\
& \quad \times \exp\left(a_+^i(q^{\frac{k+h^\vee}{2}}z) + \sum_{l=j}^N (b_+^{i,l}(q^{k+l}z) - b_+^{i+1,l}(q^{k+l-1}z))\right) : \Big). \quad (3.3)
\end{aligned}$$

Then we have the next proposition followed from [31]:

Proposition 2 . $h_{q,k}(\psi_\pm^i(z))$ and $h_{q,k}(e^{\pm i}(z))$ ($i = 1, \dots, N-1$) with $k = c$ satisfy the commutation relations (2.8)-(2.15).

As a result, when $k \neq -h^\vee$, this homomorphism $h_{q,k}$ gives the Wakimoto realization of the quantum affine algebra $U_q(\widehat{sl_N})$ with $k = c$. In the following subsections, we will construct the free field realization of the elliptic algebra $U_{q,p}(\widehat{sl_N})$ by twisting this realization.

3.2 Bosonization of elliptic currents

In this subsection, we show the first step of our construction: giving the bosonization of the elliptic currents $\Psi_i^\pm(z)$, $e_i(z)$ and $f_i(z)$ ($i = 1, \dots, N-1$) of $U_q(\widehat{sl_N})$. For brevity, in what follows we will use the same notations for the elements of $U_q(\widehat{sl_N})$ and their images in the completion of $\mathcal{H}_{q,k}$. Here we need to introduce some new currents $D_i^\pm(z; r, r^*)$ ($i = 1, \dots, N-1$) depending on parameters r and r^* as:

$$\begin{aligned}
D_i^+(z; r, r^*) &= \exp \left\{ \sum_{n>0} \frac{1}{[r^*n]} \left(\sum_{j=1}^i (b_{-n}^{j,i+1} q^{-(k+j-1)n} - b_{-n}^{j,i} q^{-(k+j)n}) + a_{-n}^i q^{-\frac{k+h^\vee}{2}n} \right. \right. \\
&\quad \left. \left. + \sum_{j=i+1}^N (b_{-n}^{j,i} q^{-(k+j)n} - b_{-n}^{j,i+1} q^{-(k+j-1)n}) \right) q^{rn} z^n \right\}, \\
D_i^-(z; r, r^*) &= \exp \left\{ - \sum_{n>0} \frac{1}{[rn]} \left(\sum_{j=1}^i (b_n^{j,i+1} q^{-(k+j-1)n} - b_n^{j,i} q^{-(k+j)n}) + a_n^i q^{-\frac{k+h^\vee}{2}n} \right. \right. \\
&\quad \left. \left. + \sum_{j=i+1}^N (b_n^{j,i} q^{-(k+j)n} - b_n^{j,i+1} q^{-(k+j-1)n}) \right) q^{rn} z^n \right\},
\end{aligned}$$

$$+ \sum_{j=i+1}^N (b_n^{i,j} q^{-(k+j)n} - b_n^{i+1,j} q^{-(k+j-1)n}) q^{rn} z^{-n} \Big\},$$

which are nominated as twisting currents; then we have the following lemma:

Lemma 1 . *The currents $D_i^\pm(z; r, r^*)$ ($i = 1, \dots, N-1$) and the fields in Eqns.(3.1)-(3.3) satisfy the following commutation relations:*

$$D_i^+(z; r, r^*) D_j^-(w; r, r^*) = \frac{(pq^{-a_{ij}-k} \frac{z}{w}; p)_\infty}{(pq^{a_{ij}-k} \frac{z}{w}; p)_\infty} \frac{(p^* q^{a_{ij}+k} \frac{z}{w}; p^*)_\infty}{(p^* q^{-a_{ij}+k} \frac{z}{w}; p^*)_\infty} D_j^-(w; r, r^*) D_i^+(z; r, r^*), \quad (3.4)$$

$$D_i^\pm(z; r, r^*) D_j^\pm(w; r, r^*) = D_j^\pm(w; r, r^*) D_i^\pm(z; r, r^*), \quad (3.5)$$

$$D_i^+(z; r, r^*) \psi_+^j(w) = \frac{(pq^{-a_{ij}-\frac{k}{2}} \frac{z}{w}; p^*)_\infty}{(pq^{a_{ij}-\frac{k}{2}} \frac{z}{w}; p^*)_\infty} \frac{(p^* q^{a_{ij}-\frac{k}{2}} \frac{z}{w}; p^*)_\infty}{(p^* q^{-a_{ij}-\frac{k}{2}} \frac{z}{w}; p^*)_\infty} \psi_+^j(w) D_i^+(z; r, r^*), \quad (3.6)$$

$$D_i^+(z; r, r^*) \psi_-^j(w) = \psi_-^j(w) D_i^+(z; r, r^*), \quad (3.7)$$

$$D_i^+(z; r, r^*) e^{+,j}(w) = \frac{(p^* q^{a_{ij}} \frac{z}{w}; p^*)_\infty}{(p^* q^{-a_{ij}} \frac{z}{w}; p^*)_\infty} e^{+,j}(w) D_i^+(z; r, r^*), \quad (3.8)$$

$$D_i^+(z; r, r^*) e^{-,j}(w) = \frac{(p^* q^{-a_{ij}+k} \frac{z}{w}; p^*)_\infty}{(p^* q^{a_{ij}+k} \frac{z}{w}; p^*)_\infty} e^{-,j}(w) D_i^+(z; r, r^*), \quad (3.9)$$

$$D_i^-(z; r, r^*) \psi_+^j(w) = \psi_+^j(w) D_i^-(z; r, r^*), \quad (3.10)$$

$$D_i^-(z; r, r^*) \psi_-^j(w) = \frac{(pq^{a_{ij}+\frac{k}{2}} \frac{w}{z}; p)_\infty}{(pq^{-a_{ij}+\frac{k}{2}} \frac{w}{z}; p)_\infty} \frac{(p^* q^{-a_{ij}+\frac{k}{2}} \frac{w}{z}; p)_\infty}{(p^* q^{a_{ij}+\frac{k}{2}} \frac{w}{z}; p)_\infty} \psi_-^j(w) D_i^-(z; r, r^*), \quad (3.11)$$

$$D_i^-(z; r, r^*) e^{+,j}(w) = \frac{(pq^{-a_{ij}-k} \frac{w}{z}; p)_\infty}{(pq^{a_{ij}-k} \frac{w}{z}; p)_\infty} e^{+,j}(w) D_i^-(z; r, r^*), \quad (3.12)$$

$$D_i^-(z; r, r^*) e^{-,j}(w) = \frac{(pq^{a_{ij}} \frac{w}{z}; p)_\infty}{(pq^{-a_{ij}} \frac{w}{z}; p)_\infty} e^{-,j}(w) D_i^-(z; r, r^*). \quad (3.13)$$

Proof: A straightforward but lengthy operator product expansions (OPE) calculation verifies this lemma. Here, we only take the first one as an example. It is obvious to see that

$$D_i^+(z; r, r^*) D_j^-(w; r, r^*) =: D_i^+(z; r, r^*) D_j^-(w; r, r^*) ::$$

and using the following formulas:

$$e^A e^B = e^{[A,B]} e^B e^A, \quad \text{if } [A, B] \text{ commute with } A \text{ and } B;$$

$$\exp\left(-\sum_{n>0} \frac{x^n}{n}\right) = 1 - x;$$

$$(1 - x)^{-1} = \sum_{n \geq 0} x^n,$$

we can prove the following relations for three cases: $j = i$, $|j - i| = 1$ and $|j - i| \geq 2$:

$$D_j^-(w; r, r^*) D_i^+(z; r, r^*) = \frac{(pq^{a_{ij}-k} \frac{z}{w}; p)_\infty}{(pq^{-a_{ij}-k} \frac{z}{w}; p)_\infty} \frac{(p^* q^{-a_{ij}+k} \frac{z}{w}; p^*)_\infty}{(p^* q^{a_{ij}+k} \frac{z}{w}; p^*)_\infty} : D_j^-(w; r, r^*) D_i^+(z; r, r^*) :,$$

then we obtain (3.4) since

$$: D_i^+(z; r, r^*) D_j^-(w; r, r^*) := D_j^-(w; r, r^*) D_i^+(z; r, r^*) : .$$

The others can be proved similarly. \square

Now twisting the free boson realization (3.1)-(3.3) of $U_q(\widehat{sl_N})$ with $D_i^\pm(z; r, r^*)$, we have free bosonic fields $\Psi_i^\pm(z)$, $e_i(z)$ and $f_i(z)$ ($i = 1, \dots, N-1$) given by

$$\Psi_i^+(z) = D_i^+(q^{\frac{k}{2}}z; r, r^*) \psi_+^i(z) D_i^-(q^{-\frac{k}{2}}z; r, r^*), \quad (3.14)$$

$$\Psi_i^-(z) = D_i^+(q^{-\frac{k}{2}}z; r, r^*) \psi_-^i(z) D_i^-(q^{\frac{k}{2}}z; r, r^*), \quad (3.15)$$

$$e_i(z) = D_i^+(z; r, r^*) e^{+,i}(z), \quad (3.16)$$

$$f_i(z) = e^{-,i}(z) D_i^-(z; r, r^*); \quad (3.17)$$

then applying the above Lemma 1 and the Proposition 2, we can obtain the following theorem:

Theorem 1 . *The fields (3.14)-(3.17) with $k = c$ satisfy the elliptic commutation relations (2.16)-(2.24) in Proposition 1.*

Proof: For example, we just prove (2.16). By (3.14),

$$\Psi_i^+(z) \Psi_j^+(w) = D_i^+(q^{\frac{k}{2}}z; r, r^*) \psi_+^i(z) D_i^-(q^{-\frac{k}{2}}z; r, r^*) \times D_j^+(q^{\frac{k}{2}}w; r, r^*) \psi_+^j(w) D_j^-(q^{-\frac{k}{2}}w; r, r^*),$$

and the Proposition 2 and (3.4)-(3.6) in Lemma 1,

$$\begin{aligned} \psi_+^i(z) \psi_+^j(w) &= \psi_+^j(w) \psi_+^i(z), \\ D_i^+(q^{\frac{k}{2}}z; r, r^*) D_j^-(q^{-\frac{k}{2}}w; r, r^*) &= \frac{(pq^{-a_{ij}}z/w; p)_\infty (p^* q^{a_{ij}+2k}z/w; p^*)_\infty}{(pq^{a_{ij}}z/w; p)_\infty (p^* q^{-a_{ij}+2k}z/w; p^*)_\infty} D_j^-(q^{-\frac{k}{2}}w; r, r^*) D_i^+(q^{\frac{k}{2}}z; r, r^*), \\ D_i^+(q^{\frac{k}{2}}z; r, r^*) \psi_+^j(w) &= \frac{(pq^{-a_{ij}}z/w; p^*)_\infty (p^* q^{a_{ij}}z/w; p^*)_\infty}{(pq^{a_{ij}}z/w; p^*)_\infty (p^* q^{-a_{ij}}z/w; p^*)_\infty} \psi_+^j(w) D_i^+(q^{\frac{k}{2}}z; r, r^*), \end{aligned}$$

we get

$$\Psi_i^+(z) \Psi_j^+(w) = \frac{(pq^{-a_{ij}}z/w; p)_\infty (pq^{a_{ij}}w/z; p)_\infty (p^* q^{a_{ij}}z/w; p^*)_\infty (p^* q^{-a_{ij}}w/z; p^*)_\infty}{(pq^{a_{ij}}z/w; p)_\infty (pq^{-a_{ij}}w/z; p)_\infty (p^* q^{-a_{ij}}z/w; p^*)_\infty (p^* q^{a_{ij}}w/z; p^*)_\infty} \Psi_j^+(w) \Psi_i^+(z);$$

moreover, since the following identity holds:

$$\frac{(pq^{-a_{ij}}z/w; p)_\infty (p^* q^{a_{ij}}z/w; p^*)_\infty}{(pq^{a_{ij}}z/w; p)_\infty (p^* q^{-a_{ij}}z/w; p^*)_\infty} = \frac{(q^{-a_{ij}}z/w; p)_\infty (q^{a_{ij}}z/w; p^*)_\infty}{(q^{a_{ij}}z/w; p)_\infty (q^{-a_{ij}}z/w; p^*)_\infty}$$

the commutation relation (2.16) is obtained:

$$\begin{aligned} \Psi_i^+(z) \Psi_j^+(w) &= \frac{(q^{-a_{ij}}z/w; p)_\infty (pq^{a_{ij}}w/z; p)_\infty (q^{a_{ij}}z/w; p^*)_\infty (p^* q^{-a_{ij}}w/z; p^*)_\infty}{(q^{a_{ij}}z/w; p)_\infty (pq^{-a_{ij}}w/z; p)_\infty (q^{-a_{ij}}z/w; p^*)_\infty (p^* q^{a_{ij}}w/z; p^*)_\infty} \Psi_j^+(w) \Psi_i^+(z) \\ &= \frac{\Theta_p(q^{-a_{ij}}z/w) \Theta_{p^*}(q^{a_{ij}}z/w)}{\Theta_p(q^{a_{ij}}z/w) \Theta_{p^*}(q^{-a_{ij}}z/w)} \Psi_j^+(w) \Psi_i^+(z). \end{aligned}$$

The commutation relations (2.17)-(2.24) can be verified in the same way. \square

Corollary 1 . $\Psi_i^\pm(z)$, $e_i(z)$ and $f_i(z)$ ($i = 1, \dots, N-1$) defined above realize the elliptic currents of $U_q(\widehat{sl_N})$ with level $k = c$.

Actually, in the $p \rightarrow 0$ limit, $\Psi_i^\pm(z)$, $e_i(z)$ and $f_i(z)$ ($i = 1, \dots, N-1$) give a new free field representation of $U_q(\widehat{sl_N})$, which is different from the one in subsection 3.1. More precisely, as $p \rightarrow 0$ (or $r \rightarrow \infty$):

$$\begin{aligned}\Psi_i^+(z) &\rightarrow \left(\psi_-^i(q^k z)\right)^{-1}, & \Psi_i^-(z) &\rightarrow \left(\psi_+^i(q^k z)\right)^{-1}, \\ e_i(z) &\rightarrow q^{-h_i} \left(\psi_-^i(q^{k/2} z)\right)^{-1} e^{+,i}(z), \\ f_i(z) &\rightarrow e^{-,i}(z) q^{h_i} \left(\psi_+^i(q^{k/2} z)\right)^{-1},\end{aligned}$$

here

$$h_i = \sum_{l=1}^i (p_b^{l,i+1} - p_b^{l,i}) + p_a^i + \sum_{l=i+1}^N (p_b^{i,l} - p_b^{i+1,l}), \quad (3.18)$$

which has a lot of useful properties. And we will discuss them and apply them in the following sections.

3.3 Free field realization of $U_{q,p}(\widehat{sl_N})$

The second step of the construction is presented in this subsection. We will construct the free boson realization of the total currents. In order to do that, we need to introduce a Heisenberg algebra \mathcal{H} generated by \hat{p}_i and \hat{q}_i ($1 \leq i \leq N-1$) such that

$$[\hat{q}_i, \hat{p}_j] = \frac{a_{ij}}{2}$$

and they commute with a^i ($1 \leq i \leq N-1$), b^{ij} and c^{ij} ($1 \leq i < j \leq N$).

With them we define the fields $H_i^\pm(u)$, $E_i(u)$ and $F_i(u)$ ($i = 1, \dots, N-1$) by

$$H_i^\pm(u) = \Psi_i^\pm(z) e^{2\hat{q}_i} q^{\mp h_i} (q^{\pm(r-\frac{k}{2})} z)^{\frac{(\hat{p}_i + h_i - 1)}{r} - \frac{(\hat{p}_i - 1)}{r^*}}, \quad (3.19)$$

$$E_i(u) = e_i(z) e^{2\hat{q}_i} z^{-\frac{(\hat{p}_i - 1)}{r^*}}, \quad (3.20)$$

$$F_i(u) = f_i(z) z^{\frac{(h_i + \hat{p}_i - 1)}{r}}, \quad (3.21)$$

where h_i is given by (3.18). Then (3.19)-(3.21) define a homomorphism from $U_{q,p}(\widehat{sl_N})$ to $\widehat{\mathcal{H}}_{q,k} \otimes \mathcal{H}$. Here we have the following lemma about the h'_i s:

Lemma 2 . The following commutation relations between h_i and $\Psi_i^\pm(z)$, $e_i(z)$ and $f_i(z)$ in Eqns. (3.14)-(3.17) for $i = 1, \dots, N-1$ hold:

$$\begin{aligned}[h_i, \Psi_j^\pm(z)] &= 0, \\ [h_i, e_j(z)] &= a_{ij} e_j(z), \\ [h_i, f_j(z)] &= -a_{ij} f_j(z).\end{aligned}$$

This lemma can be easily verified by using the Hausdorff formula; and they are the useful properties that the h'_i s possess, which we mentioned at the end of the above subsection. Then we can apply this lemma and the Theorem 1 to obtain the results below:

Theorem 2 . *The fields in Eqns.(3.19)-(3.21) with $k = c$ obey the commutation relations given by (2.25)-(2.33).*

Corollary 2 . *$H_i^\pm(u)$, $E_i(u)$ and $F_i(u)$ ($i = 1, \dots, N-1$) defined above give the free boson realization of $U_{q,p}(\widehat{sl_N})$ with given level $k = c$.*

4 Screening currents

In free fields approach, one has to discuss two necessary ingredients: screening currents and vertex operators. We will only consider the screening currents of the elliptic quantum algebra $U_{q,p}(\widehat{sl_N})$ in this section. In 2D CFT, screening current is a primary field of the energy-momentum tensor with conformal weight 1, and its integration gives the screening charge. It has the property that it commutes with the currents modulo a total differential of certain field. This property ensures that the screening charge may be inserted in the correlators by changing their conformal charges without affecting their conformal properties. In this section, using the bosons a^i ($1 \leq i \leq N-1$), b^{ij} and c^{ij} ($1 \leq i < j \leq N$), we will construct a series of screening currents $S^i(z)$ ($1 \leq i \leq N-1$) of $U_{q,p}(\widehat{sl_N})$. These currents commute with the currents modulo a total q-difference of some fields, so they could be regarded as a quantum deformation of the screening currents in 2D CFT.

We denote a sort of q-difference operator with a parameter $n \in \mathbb{Z}_{>0}$ by

$${}_n\partial_z X(z) \equiv \frac{X(q^n z) - X(q^{-n} z)}{(q - q^{-1})z},$$

which is called a total q-difference of a function $X(z)$. Moreover, to eliminate the total q-difference, one can define the Jackson integral as

$$\int_0^{s\infty} X(z) d_p z \equiv s(1-p) \sum_{n \in \mathbb{Z}} X(sp^n) p^n$$

for a scalar $s \in \mathbb{C} \setminus \{0\}$ and a complex number p such that $|p| < 1$. So that,

$$\int_0^{s\infty} ({}_n\partial_z X(z)) d_p z = 0,$$

if it is convergent and we take $p = q^{2n}$.

For simplicity, we set boson fields $A_{\pm}^i(L_1, \dots, L_s; M_1, \dots, M_{s+1}|z; \alpha)$ ($i = 1, \dots, N-1$) for $\alpha \in \mathbb{C}$ with parameters L_i and M_j ($i, j \in \mathbb{N}$) as follows:

$$\begin{aligned} A_+^i(L_1, \dots, L_s; M_1, \dots, M_{s+1}|z; \alpha) &= \sum_{n>0} \frac{[L_1 n] \cdots [L_s n]}{[M_1 n] \cdots [M_{s+1} n]} a_n^i (q^\alpha z)^{-n}, \\ A_-^i(L_1, \dots, L_s; M_1, \dots, M_{s+1}|z; \alpha) &= \sum_{n>0} \frac{[L_1 n] \cdots [L_s n]}{[M_1 n] \cdots [M_{s+1} n]} a_{-n}^i (q^\alpha z)^n, \end{aligned}$$

then in terms of these boson fields and the ones introduced before, we express the screening currents $S^i(z)$ ($i = 1, \dots, N-1$) as:

$$\begin{aligned} S^i(z) &= \frac{-1}{(q - q^{-1})z} : \exp \left\{ A_-^i \left(-(k + h^\vee) | z; -\frac{k + h^\vee}{2} \right) + A_+^i \left(k + h^\vee | z; \frac{k + h^\vee}{2} \right) \right. \\ &\quad \left. - \frac{1}{k + h^\vee} (q_a^i + p_a^i \ln z) \right\} : \\ &\quad \times \left\{ \sum_{j=i+1}^N : \exp \left((b + c)^{i+1, j} (q^{N-j} z) \right) \right. \\ &\quad \times \left(\exp(-b_-^{i, j} (q^{N-j} z) - (b + c)^{i, j} (q^{N-j+1} z)) \right. \\ &\quad \left. - \exp(-b_+^{i, j} (q^{N-j} z) - (b + c)^{i, j} (q^{N-j-1} z)) \right) \\ &\quad \left. \times \exp \left(\sum_{l=j+1}^N (b_-^{i+1, l} (q^{N-l+1} z) - b_-^{i, l} (q^{N-l} z)) \right) : \right\}, \end{aligned}$$

and they possess the following properties:

Theorem 3 : $S^i(z)$ and the currents $H_i^\pm(z)$, $E_i(z)$ and $F_i(z)$ ($i = 1, \dots, N-1$) in Eqns.(3.19)-(3.21) satisfy the relations:

$$\begin{aligned} H_i^\pm(z) S^j(w) &= S^j(w) H_i^\pm(z) = O(1), \\ E_i(z) S^j(w) &= S^j(w) E_i(z) = O(1), \\ F_i(z) S^j(w) &= S^j(w) F_i(z) = \delta^{ij}_{(k+h^\vee)} \partial_w \left[\frac{1}{z-w} \tilde{S}^i(z) \right] + O(1), \\ S^i(z) S^j(w) &= \frac{\theta_{k+h^\vee}(u - v + \frac{a_{ij}}{2})}{\theta_{k+h^\vee}(u - v - \frac{a_{ij}}{2})} S^j(w) S^i(z), \end{aligned}$$

where the symbol $O(1)$ means regularity and $\tilde{S}^i(z)$ ($i = 1, \dots, N-1$) are given by:

$$\begin{aligned} \tilde{S}^i(z) &=: \exp \left\{ A_-^i \left(-(k + h^\vee) | z; \frac{k + h^\vee}{2} \right) + A_+^i \left(k + h^\vee | z; -\frac{k + h^\vee}{2} \right) \right. \\ &\quad \left. - \frac{1}{k + h^\vee} (q_a^i + p_a^i \ln z) \right\} D_i^-(z; r, r^*) z^{\frac{(h_i + \hat{p}_i - 1)}{r}} : . \end{aligned}$$

It is obvious to note that since these screening currents do not contain the parameter p , they are also the screening currents of the quantum affine algebra $U_q(\widehat{sl_N})$. As a result, the above theorem can be easily proved by applying the results in [31]. Once we give the explicit expressions of the screening currents, we can calculate the cohomology and study the irreducibility of modules of the algebra, which we will discuss separately in the future.

5 Vertex operators

In this section, except for the screening currents, we will study the other important object that one has to discuss in free fields approach: vertex operators (VOs) of $U_{q,p}(\widehat{sl_N})$. In WZW model, the primary fields could be realized as the highest weight representation of Kac-Moody algebra, which are commonly called as vertex operators (VOs) or intertwining operators. For quantum affine algebra, in [15], the authors defined q -deformed VOs as certain intertwining operators, which could be regarded as the quantum counterpart of the primary field in 2D CFT. They play crucial roles in calculating correlation functions. Following this approach, in this section we will construct the free field realization of the VOs of $U_{q,p}(\widehat{sl_N})$. There are two types of them: the type *I* VOs and the type *II* VOs. They can all be viewed as the elliptic analogs of the primary fields. The explicit expressions of them are obtained by twisting the corresponding ones of the quantum affine algebra $U_q(\widehat{sl_N})$, in which the type *II* VOs is not given before. In fact, even for the classical affine Lie algebras, the type *II* VOs are not given. In this section, we will give the type *II* VOs of $U_q(\widehat{sl_N})$ and then use it to construct the one of $U_{q,p}(\widehat{sl_N})$.

5.1 The type *I* and type *II* VOs of $U_q(\widehat{sl_N})$

In this subsection, we first review the primary field of the quantum affine algebra $U_q(\widehat{sl_N})$ given in [31], which is the type *I* VOs of it. Here we denote it as $\phi_{\vec{\Lambda}}(z)$ with $\vec{\Lambda} = (\lambda^1, \dots, \lambda^{N-1})$, where $\vec{\Lambda}$ is the weight of the classical affine Lie algebra. However, we will reexpress it by using some new bosons $\{\check{a}^i : i = 1, \dots, N-1\}$ defined as:

$$\check{a}_n^i = \sum_{j=1}^{N-1} n \frac{[\min(i, j)n][N - \max(i, j)]}{[(k + h^\vee)n][Nn][n]^2} a_n^j$$

for any $n \in \mathbb{Z}_{\neq 0}$, and the zero-modes are

$$\check{p}_a^i = \sum_{j=1}^{N-1} \frac{\min(i, j)(N - \max(i, j))}{(k + h^\vee)N} p_a^j,$$

$$\check{q}_a^i = \sum_{j=1}^{N-1} \frac{\min(i, j)(N - \max(i, j))}{(k + h^\vee)N} q_a^j,$$

here these bosons can be called the dual bosons of the original ones $\{a^i : i = 1, \dots, N-1\}$ in the sense that they satisfy the relations below:

$$[\check{a}_n^i, a_m^j] = \delta_{i,j} \delta_{n+m,0},$$

$$[\check{p}_a^i, q_a^j] = \delta_{i,j},$$

$$[\check{q}_a^i, p_a^j] = -\delta_{i,j},$$

and these relations can be verified easily by using the q -analog of the inverse of the Cartan matrix:

$$\sum_{r=1}^{N-1} \frac{[a_{i,r}n][\min(r, j)n][(N - \max(r, j))n]}{[Nn][n]^2} = \delta_{i,j}. \quad (5.22)$$

With them, the type I VOs $\phi_{\vec{\lambda}}(z)$ with weight $\vec{\lambda} = (\lambda^1, \dots, \lambda^{N-1})$ of the algebra $U_q(\widehat{sl_N})$ can be rewritten as:

$$\phi_{\vec{\lambda}}(z) =: \prod_{i=1}^{N-1} \phi_{\lambda^i}(z) :,$$

here the fields $\phi_{\lambda^i}(z)$ for $i = 1, \dots, N-1$, which are called the components of $\phi_{\vec{\lambda}}(z)$, are given by:

$$\phi_{\lambda^i}(z) = \exp \left\{ - \sum_{n>0} \frac{[\lambda^i n]}{n} \check{a}_{-n}^i q^{\frac{k+h^\vee}{2}n} z^n \right\} \exp \left\{ \lambda^i (\check{q}_a^i + \check{p}_a^i \ln z) \right\} \exp \left\{ - \sum_{n>0} \frac{[\lambda^i n]}{n} \check{a}_n^i q^{\frac{k+h^\vee}{2}n} z^{-n} \right\};$$

and it has the following properties with the fields $\psi_{\pm}^i(z)$ and $e^{\pm i}(z)$ in Eqns.(3.1)-(3.3) for $i = 1, \dots, N-1$ proved in [31] :

$$\psi_{\pm}^i(z) \phi_{\vec{\lambda}}(w) = q^{-\lambda^i} \frac{w - q^{\lambda^i \mp \frac{k}{2}} z}{w - q^{-\lambda^i \mp \frac{k}{2}} z} \phi_{\vec{\lambda}}(w) \psi_{\pm}^i(z); \quad (5.23)$$

$$e^{+,i}(z) \phi_{\vec{\lambda}}(w) = \phi_{\vec{\lambda}}(w) e^{+,i}(z); \quad (5.24)$$

$$e^{-,i}(z) \phi_{\vec{\lambda}}(w) = q^{-\lambda^i} \frac{w - q^{\lambda^i} z}{w - q^{-\lambda^i} z} \phi_{\vec{\lambda}}(w) e^{-,i}(z). \quad (5.25)$$

For any weight $\vec{\lambda} = (\lambda^1, \dots, \lambda^{N-1})$, the type II VOs $\psi_{\vec{\lambda}}(z)$ of $U_q(\widehat{sl_N})$ is not known before. We present them in terms of $\{b^{i,j}, c^{i,j} : 1 \leq i < j \leq N-1\}$ and the above mentioned dual bosons. The field $\psi_{\vec{\lambda}}(z)$ can be expressed as:

$$\psi_{\vec{\lambda}}(z) =: \prod_{i=1}^{N-1} \psi_{\lambda^i}(z) :$$

here its components $\psi_{\lambda^i}(z)$ are defined as:

$$\begin{aligned} \psi_{\lambda^i}(z) = & \exp \left\{ - \sum_{n>0} \frac{[\lambda^{N-i}n]}{n} \check{a}_{-n}^i q^{-\frac{k+h^\vee}{2}n} z^n \right\} \exp \left\{ \lambda^{N-i} (\check{q}_a^i + \check{p}_a^i \ln z) \right\} \exp \left\{ - \sum_{n>0} \frac{[\lambda^{N-i}n]}{n} \check{a}_n^i q^{-\frac{k+h^\vee}{2}n} z^{-n} \right\} \\ & \times \exp \left\{ \sum_{n>0} \sum_{j=i+1}^N \frac{[\lambda^{j-i}n]}{[n]^2} (b_{-n}^{i,j} + c_{-n}^{i,j}) z^n \right\} \exp \left\{ \sum_{j=i+1}^N \lambda^{j-i} (q_b^{i,j} + q_c^{i,j} + (p_b^{i,j} + p_c^{i,j}) \ln z) \right\} \\ & \times \exp \left\{ - \sum_{n>0} \sum_{j=i+1}^N \frac{[\lambda^{j-i}n]}{[n]^2} (b_n^{i,j} + c_n^{i,j}) z^{-n} \right\}. \end{aligned}$$

Furthermore, we have also proved the following theorem:

Theorem 4 . *The field $\psi_{\vec{\lambda}}(z)$ satisfies the intertwining relations:*

$$\begin{aligned} \psi_{\pm}^i(z) \psi_{\vec{\lambda}}(w) &= q^{\lambda^i} \frac{w - q^{-\lambda^i \pm \frac{k}{2}} z}{w - q^{\lambda^i \pm \frac{k}{2}} z} \psi_{\vec{\lambda}}(w) \psi_{\pm}^i(z), \\ e^{+,i}(z) \psi_{\vec{\lambda}}(w) &= q^{\lambda^i} \frac{w - q^{-\lambda^i} z}{w - q^{\lambda^i} z} \psi_{\vec{\lambda}}(w) e^{+,i}(z), \\ e^{-,i}(z) \psi_{\vec{\lambda}}(w) &= \psi_{\vec{\lambda}}(w) e^{-,i}(z), \end{aligned}$$

where $\psi_{\pm}^i(z)$ and $e^{\pm,i}(z)$ for $i = 1, \dots, N-1$ are the currents given by (3.1)-(3.3).

Proof: Here we only list the useful formulas we used to prove this theorem:

$$\begin{aligned} e^A B &= e^{[A,B]} B e^A, \text{ if } [A, B] \text{ is a constant}; \\ e^A e^B &= e^{[A,B]} e^B e^A, \text{ if } [A, B] \text{ commute with } A \text{ and } B; \\ \exp \left(- \sum_{n>0} \frac{x^n}{n} \right) &= 1 - x; \\ (1 - x)^{-1} &= \sum_{n \geq 0} x^n; \end{aligned}$$

and the q-analog of the inverse of the Cartan matrix in (5.22) is also used. \square

These intertwining relations could be used to characterize the type II VOs of $U_q(\widehat{sl_N})$. Lastly, we present the commutation relations among the type I VOs $\phi_{\vec{\lambda}}(z)$ and type II VOs $\psi_{\vec{\lambda}}(z)$; and here we only compute the commutation relations between their components.

Proposition 3 :

$$\begin{aligned} \phi_{\lambda^i}(z) \phi_{\lambda^j}(w) &= \left(\frac{z}{w} \right)^{\lambda^i \lambda^j g^{i,j}} \exp \left\{ X_1 \left(\frac{z}{w} \right) \right\} \exp \left\{ -X_1(z \leftrightarrow w) \right\} \phi_{\lambda^j}(w) \phi_{\lambda^i}(z); \\ \phi_{\lambda^i}(z) \psi_{\lambda^j}(w) &= \left(\frac{z}{w} \right)^{\lambda^i \lambda^{N-j} g^{i,j}} \exp \left\{ X_2 \left(\frac{z}{w} \right) \right\} \exp \left\{ -X_2(z \leftrightarrow w) \right\} \psi_{\lambda^j}(w) \phi_{\lambda^i}(z); \\ \psi_{\lambda^i}(z) \psi_{\lambda^j}(w) &= \left(\frac{z}{w} \right)^{\lambda^{N-i} \lambda^{N-j} g^{i,j}} \exp \left\{ X_3 \left(\frac{z}{w} \right) \right\} \exp \left\{ -X_3(z \leftrightarrow w) \right\} \psi_{\lambda^j}(w) \psi_{\lambda^i}(z), \end{aligned}$$

here the $X_i\left(\frac{z}{w}\right)$ for $i = 1, 2, 3$ are given by

$$\begin{aligned} X_1\left(\frac{z}{w}\right) &= \sum_{n>0} \frac{1}{n} \frac{[\lambda^i n][\lambda^j n]}{[n]^2} [g_n^{i,j}] q^{(k+h^\vee)n} \left(\frac{z}{w}\right)^n; \\ X_2\left(\frac{z}{w}\right) &= \sum_{n>0} \frac{1}{n} \frac{[\lambda^i n][\lambda^{N-j} n]}{[n]^2} [g_n^{i,j}] \left(\frac{z}{w}\right)^n; \\ X_3\left(\frac{z}{w}\right) &= \sum_{n>0} \frac{1}{n} \frac{[\lambda^{N-i} n][\lambda^{N-j} n]}{[n]^2} [g_n^{i,j}] q^{-(k+h^\vee)n} \left(\frac{z}{w}\right)^n, \end{aligned}$$

where for simplicity we use the symbols $g^{i,j}$ and $[g_n^{i,j}]$ to denote:

$$g^{i,j} = \frac{\min(i, j)(N - \max(i, j))}{(k + h^\vee)N}, \quad (5.26)$$

$$[g_n^{i,j}] = \frac{[\min(i, j)n][(N - \max(i, j))n]}{[(k + h^\vee)n][Nn]}. \quad (5.27)$$

It should be remarked that the matrix $G = (g^{i,j})_{1 \leq i, j \leq N-1}$ is the inverse matrix of $(k + h^\vee)A$, and here A is the Cartan matrix.

5.2 The type I and type II VOs of $U_{q,p}(\widehat{sl_N})$

In this subsection, we will give the free field realization of the type I and type II VOs of $U_{q,p}(\widehat{sl_N})$. We nominate them as $\Phi_{\vec{\lambda}}(u)$ and $\Psi_{\vec{\lambda}}(u)$ with weight $\vec{\lambda} = (\lambda^1, \dots, \lambda^{N-1})$. They are all obtained by twisting the corresponding ones of $U_q(\widehat{sl_N})$ given in the above subsection.

First, we will construct two twisted currents $T_{\pm}(z; p)$ which depend on the parameter p for the two types VOs of $U_q(\widehat{sl_N})$. For the type I VOs, we define the twisted current $T_+(z; p)$ as

$$T_+(z; p) =: \prod_{i=1}^{N-1} T_+^i(z; p) :$$

and for $i = 1, \dots, N-1$,

$$T_+^i(z; p) = \exp \left\{ \sum_{n>0} \frac{1}{n} \frac{[\lambda^i n][kn]}{[rn]} \check{H}_n^i q^{(r-\frac{k}{2})n} z^{-n} \right\} \exp \left\{ -\frac{\lambda^i}{r} (\check{p}_i + \check{h}_i - (k + h^\vee) \check{p}_a^i) \ln z \right\}$$

here

$$\check{H}_n^i = \sum_{j=1}^{N-1} n \frac{[\min(i, j)n][(N - \max(i, j))n]}{[kn][Nn][n]^2} H_n^j, \quad \forall n \in \mathbb{Z}_{\neq 0} \quad (5.28)$$

and

$$H_n^i = \sum_{j=1}^i \left(b_n^{j,i+1} q^{-(\frac{k}{2}+j-1)|n|} - b_n^{j,i} q^{-(\frac{k}{2}+j)|n|} \right) + a_n^i q^{-\frac{k}{2}|n|} + \sum_{j=i+1}^N \left(b_n^{i,j} q^{-(\frac{k}{2}+j)|n|} - b_n^{i+1,j} q^{-(\frac{k}{2}+j-1)|n|} \right),$$

then it is easy to verify that the following relation holds:

$$[\check{H}_n^i, H_m^j] = \delta_{i,j} \delta_{n+m,0},$$

since we have the following commutation relation

$$[H_n^i, H_m^j] = \frac{[a_{ij}n][kn]}{n} \delta_{n+m,0};$$

moreover, the symbols \check{h}_i and \check{p}_i are used to denote the following complicated ones:

$$\check{h}_i = \sum_{j=1}^{N-1} \frac{\min(i,j)(N - \max(i,j))}{N} h_j,$$

$$\check{p}_i = \sum_{j=1}^{N-1} \frac{\min(i,j)(N - \max(i,j))}{N} \hat{p}_j,$$

and \check{p}_a^i was defined at the beginning of the above subsection. And for the type *II* VOs, the twisted current $T_-(z; p)$ is given by

$$T_-(z; p) =: \prod_{i=1}^{N-1} T_-^i(z; p) :$$

and

$$T_-^i(z; p) = \exp \left\{ \sum_{n>0} \frac{1}{n} \frac{[\lambda^i n][kn]}{[r^* n]} \check{H}_{-n}^i q^{(r-\frac{k}{2})n} z^n \right\} \exp \left\{ -2\lambda^i \check{q}_i + \frac{\lambda^i}{r^*} \check{p}_i \ln z \right\},$$

here \check{H}_{-n}^i for $n > 0$ is given in (5.28) and the symbol \check{q}_i is defined as follow:

$$\check{q}_i = \sum_{j=1}^{N-1} \frac{\min(i,j)(N - \max(i,j))}{N} \hat{q}_j.$$

Next, we can use the components $T_{\pm}^i(z; p)$ of the twisted currents to twist the ones $\phi_{\lambda^i}(z)$ and $\psi_{\lambda^i}(z)$ as follows:

$$\Phi_{\lambda^i}(u) =: \phi_{\lambda^i}(z) T_+^i(z; p) :,$$

$$\Psi_{\lambda^i}(u) =: T_-^i(z; p) \psi_{\lambda^i}(z) :,$$

here $\Phi_{\lambda^i}(u)$ and $\Psi_{\lambda^i}(u)$ could also be considered as the components of the fields $\Phi_{\vec{\lambda}}(u)$ and $\Psi_{\vec{\lambda}}(u)$, since we define $\Phi_{\vec{\lambda}}(u)$ and $\Psi_{\vec{\lambda}}(u)$ by:

$$\Phi_{\vec{\lambda}}(u) =: \prod_{i=1}^{N-1} \Phi_{\lambda^i}(u) :,$$

$$\Psi_{\vec{\lambda}}(u) =: \prod_{i=1}^{N-1} \Psi_{\lambda^i}(u) :.$$

Furthermore, we obtain an important theorem by applying the relations (5.23)-(5.25) and the Theorem 4:

Theorem 5 . *The fields $\Phi_{\vec{\lambda}}(u)$ and $\Psi_{\vec{\lambda}}(u)$ with given weight $\vec{\lambda} = (\lambda^1, \dots, \lambda^{N-1})$ possess the intertwining properties:*

$$H_i^\pm(u) \Phi_{\vec{\lambda}}(v) = \frac{\theta_r(u - v + \frac{\lambda^i}{2} \mp \frac{k}{4})}{\theta_r(u - v - \frac{\lambda^i}{2} \mp \frac{k}{4})} \Phi_{\vec{\lambda}}(v) H_i^\pm(u), \quad (5.29)$$

$$E_i(u) \Phi_{\vec{\lambda}}(v) = \Phi_{\vec{\lambda}}(v) E_i(u), \quad (5.30)$$

$$F_i(u) \Phi_{\vec{\lambda}}(v) = \frac{\theta_r(u - v + \frac{\lambda^i}{2})}{\theta_r(u - v - \frac{\lambda^i}{2})} \Phi_{\vec{\lambda}}(v) F_i(u); \quad (5.31)$$

$$H_i^\pm(u) \Psi_{\vec{\lambda}}(v) = \frac{\theta_{r^*}(u - v - \frac{\lambda^i}{2} \pm \frac{k}{4})}{\theta_{r^*}(u - v + \frac{\lambda^i}{2} \pm \frac{k}{4})} \Psi_{\vec{\lambda}}(v) H_i^\pm(u), \quad (5.32)$$

$$E_i(u) \Psi_{\vec{\lambda}}(v) = \frac{\theta_{r^*}(u - v - \frac{\lambda^i}{2})}{\theta_{r^*}(u - v + \frac{\lambda^i}{2})} \Psi_{\vec{\lambda}}(v) E_i(u), \quad (5.33)$$

$$F_i(u) \Psi_{\vec{\lambda}}(v) = \Psi_{\vec{\lambda}}(v) F_i(u), \quad (5.34)$$

where $H_i^\pm(u)$, $E_i(u)$ and $F_i(u)$ ($i = 1, \dots, N - 1$) are the total currents in (3.19)-(3.21).

The above relations (5.29)-(5.34) could be used to define the VOs of the elliptic quantum algebra $U_{q,p}(\widehat{sl_N})$. As a result, we actually gave the free field realization of the type *I* and type *II* VOs of $U_{q,p}(\widehat{sl_N})$ with given level k . Lastly, we also investigate the commutation relations among the VOs $\Phi_{\vec{\lambda}}(u)$ and $\Psi_{\vec{\lambda}}(u)$:

Proposition 4 . *For the components $\Phi_{\lambda^i}(z)$ and $\Psi_{\lambda^i}(z)$ of the type *I* VOs and type *II* VOs, we have the following relations:*

$$\Phi_{\lambda^i}(z) \Phi_{\lambda^j}(w) = \left(\frac{z}{w} \right)^{\lambda^i \lambda^j g^{i,j}} \exp \left\{ Y_1 \left(\frac{z}{w} \right) \right\} \exp \left\{ -Y_1(z \leftrightarrow w) \right\} \Phi_{\lambda^j}(w) \Phi_{\lambda^i}(z),$$

$$\Phi_{\lambda^i}(z) \Psi_{\lambda^j}(w) = \left(\frac{z}{w} \right)^{\lambda^i \lambda^{N-j} g^{i,j}} \exp \left\{ -\frac{\lambda^i(k+g)}{r} (\lambda^j g^{i,j} + C^{i,j}) \ln z \right\}$$

$$\begin{aligned}
& \times \exp \left\{ \left(X_2 + Y_2 + Y_3 + Y_4 \right) \left(\frac{z}{w} \right) \right\} \exp \left\{ -X_2(z \leftrightarrow w) \right\} \Psi_{\lambda^i(w)} \Phi_{\lambda^i}(z), \\
\Psi_{\lambda^i}(z) \Psi_{\lambda^j}(w) &= \left(\frac{z}{w} \right)^{\lambda^{N-i} \lambda^{N-j} g^{i,j}} \exp \left\{ \left(X_3 + Y_5 + Y_6 \right) \left(\frac{z}{w} \right) \right\} \\
& \times \exp \left\{ - \left(X_3 + Y_5 + Y_6 \right) (z \leftrightarrow w; i \leftrightarrow j) \right\} \Psi_{\lambda^j}(w) \Psi_{\lambda^i}(z),
\end{aligned}$$

here $g^{i,j}$ is given in (5.26); and the symbols $C^{i,j}$, $\{Y_i(\frac{z}{w}) : i = 1, \dots, 6\}$ are used to simplify the complicated ones given below:

$$\begin{aligned}
C^{i,j} &= - \sum_{l=j}^{N-1} \lambda^{l+1-j} g^{i,l} + \sum_{l=j+1}^{N-1} \lambda^{l-j} g^{i,l} - \left(\sum_{l=j+1}^N \lambda^{l-j} \right) g^{i,j} + \left(\sum_{l=j+1}^N \lambda^{l-j} \right) g^{i,j-1}; \\
Y_1\left(\frac{z}{w}\right) &= - \sum_{n>0} \frac{1}{n} \frac{[\lambda^i n][\lambda^j n][(r-k-h^\vee)n]}{[rn][n]^2} [g_n^{i,j}] q^{(r+k+h^\vee)n} \left(\frac{z}{w}\right)^n, \\
Y_2\left(\frac{z}{w}\right) &= - \sum_{n>0} \frac{1}{n} \frac{[\lambda^i n][\lambda^j n][(k+h^\vee)n]}{[rn][n]^2} [g_n^{i,j}] q^{(r-k)n} \left(\frac{z}{w}\right)^n, \\
Y_3\left(\frac{z}{w}\right) &= - \sum_{n>0} \frac{1}{n} \frac{[\lambda^i n][\lambda^{N-j} n][(k+h^\vee)n]}{[rn][n]^2} [g_n^{i,j}] q^{(r-k-h^\vee)n} \left(\frac{z}{w}\right)^n, \\
Y_4\left(\frac{z}{w}\right) &= - \sum_{n>0} \frac{1}{n} \frac{[\lambda^i n][(k+h^\vee)n]}{[rn][n]^2} [g_n^{i,j}] [C^{i,j}] q^{(r-k/2)n} \left(\frac{z}{w}\right)^n, \\
Y_5\left(\frac{z}{w}\right) &= - \sum_{n>0} \frac{1}{n} \frac{[\lambda^{N-i} n][\lambda^j n][(k+h^\vee)n]}{[r^* n][n]^2} [g_n^{i,j}] q^{(r-k-h^\vee)n} \left(\frac{z}{w}\right)^n, \\
Y_6\left(\frac{z}{w}\right) &= - \sum_{n>0} \frac{1}{n} \frac{[\lambda^j n][(k+h^\vee)n]}{[r^* n][n]^2} [g_n^{i,j}] ([C^{i,j}](i \leftrightarrow j)) q^{(r-k/2)n} \left(\frac{z}{w}\right)^n,
\end{aligned}$$

where $[g_n^{i,j}]$ is given by (5.27) and $[C^{i,j}]$ is defined by:

$$\begin{aligned}
[C^{i,j}] &= - \left(\sum_{l=j}^{N-1} [\lambda^{l+1-j} n] [g_n^{i,l}] \right) q^{-(k/2+j-1)n} + \left(\sum_{l=j+1}^{N-1} [\lambda^{l-j} n] [g_n^{i,l}] \right) q^{-(k/2+j)n} \\
&\quad - \left(\sum_{l=j+1}^N [\lambda^{l-j} n] q^{-(k/2+l)n} \right) [g_n^{i,j}] + \left(\sum_{l=j+1}^N [\lambda^{l-j} n] q^{-(k/2+l-1)n} \right) [g_n^{i,j-1}].
\end{aligned}$$

6 Discussion

In this paper, we construct the free field representation of $U_{q,p}(\widehat{sl_N})$ with given level k by twisting the Wakimoto realization of the quantum affine algebra $U_q(\widehat{sl_N})$. The free boson realization of its

screening currents are also given. Moreover, the explicit expressions of the type *II* VOs of $U_q(\widehat{sl_N})$ and the two types VOs of $U_{q,p}(\widehat{sl_N})$ are presented. In fact, even for the classical affine Lie algebras, the type *II* VOs are not given. We also have much interests in the derivation of the multi-point correlation functions, but in view of its complexity and the length of the manuscript, it will be discussed in the future. Meanwhile, it is also very interesting to extend our results to other types of Lie algebras, and we will discuss them in a separate paper.

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